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(Nov-Dec)

FYUP : 3rd Semester Examination

MAJOR

MATHEMATICS

(Group Theory)

MTH–201

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **four** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Let $G = \{A \in M_2(\mathbb{R}) \mid \det(A) = 1\}$. Prove that G forms a non-Abelian group under matrix multiplication. 5

(b) Explain why $\mathbb{Z}_8$ does not form a group under multiplication modulo 8. 2

(c) Let $\mathbb{C}^* = \{a + ib \mid a, b \in \mathbb{R},\ a^2 + b^2 \neq 0\}$ be the group of non-zero complex numbers under multiplication. Find the inverse of an element $a + ib$ in $\mathbb{C}^*$. 3

(d) Prove that G is an Abelian group if and only if $(ab)^2 = a^2b^2$ for all $a, b \in G$. 4

(e) Prove that if every element of a group G is its own inverse, then G is Abelian. 4

2. (a) Prove that the set of all positive rational numbers forms an Abelian group under the operation defined by $a * b = \frac{ab}{2}$. 4

(b) Let G be a group, and $a, b, c \in G$. Prove that $ab = ac$ implies $b = c$. 2

(c) Determine the order of every element of $\mathbb{Z}_6$ under addition modulo 6. 3

(d) Let G be a group and $a \in G$ such that $\text{ord}(a) = n$. If $a^k = e$ for some integer k, then prove that n divides k. 3

(e) Give an example of an infinite group containing an element of finite order. 2

(f) Prove that in any group, an element and its inverse have the same order. 4

UNIT—II

3. (a) Let S_n denote the set of all permutations on the set $A = \{1, 2, 3, \cdots, n\}$, where $n \geq 3$. Prove that—

   (i) S_n forms a group under composition of maps;

   (ii) S_n is non-Abelian. 3+2=5

(b) Find all subgroups of S_3. 3

(c) What is the inverse of a k-cycle in S_n? Prove that the order of a k-cycle is k. 3+3=6

(d) Let $\alpha = (12345)(678)$ and $\beta = (23847)(56)$ be permutations in S_8. Find $\alpha^2\beta$ and $\beta^{-1}\alpha$. 5

4. (a) Prove that disjoint cycles in S_n commute. 4

(b) Prove that every permutation in S_n can be expressed as a product of transpositions. Is the representation, of a permutation as a product of transpositions, unique? Justify your answer. 4+1=5

(c) Determine whether the following permutations are odd or even : 3

   (i) (12) (345) (789)

   (ii) (62351) (7249)

(d) Prove that A_n, the set of even permutations in S_n, is a subgroup of S_n. 4

(e) Show that the product of any two odd permutations is an even permutation. 3

UNIT—III

5. (a) Prove that every cyclic group is Abelian. 3

(b) If a is a generator of a group G, prove that a^{-1} is also a generator of G. 4

(c) Let G be a group and $a \in G$. Define the centralizer of a in G, denoted by $C(a)$. Prove that $C(a)$ is a subgroup of G. 1+3=4

(d) Prove that every subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$, for some integer n. 4

(e) Find all proper subgroups of the quaternion group

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

Are all of them cyclic? 4

6. (a) Let H be a subgroup of G and $a, b \in G$. Prove that $aH = bH$ if and only if $a^{-1}b \in H$. 4

(b) Determine the index of $3\mathbb{Z}$ in $\mathbb{Z}$. 3

(c) Let G be a finite group and $a \in G$. Prove that $\text{ord}(a) = |\langle a \rangle|$, and further show that $\text{ord}(a)$ divides $|G|$. 5

(d) Prove that an infinite cyclic group has exactly two generators. 3

(e) State and prove Fermat's Little theorem. 4

UNIT—IV

7. (a) Prove that the intersection of any two normal subgroups of G is normal in G. 4

(b) If H is a subgroup of index 2 in G, prove that H is normal in G. Explain why A_3 is a normal subgroup of S_3. 4+2=6

(c) Determine, up to isomorphism, all quotient groups of $\mathbb{Z}_{12}$. 5

(d) Let $\phi : G \to G'$ be a group homomorphism. Prove that ϕ is one-one if and only if $\ker \phi = \{e\}$. 4

8. (a) State and prove the fundamental theorem of group homomorphism. 1+5=6

(b) Let G be a group and $a \in G$. Define a map $f_a : G \to G$ as $f_a(x) = a^{-1}xa$ for all $x \in G$. Prove that f_a is an automorphism. 4

(c) Find all automorphisms of $\mathbb{Z}_6$. 4

(d) Prove that any non-cyclic group of order 4 is isomorphic to the Klein 4-group. 5

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